The Darwin Instability

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Literature
5.1, *Evolutionary Processes in Binary and Multiple Stars*, P. Eggleton

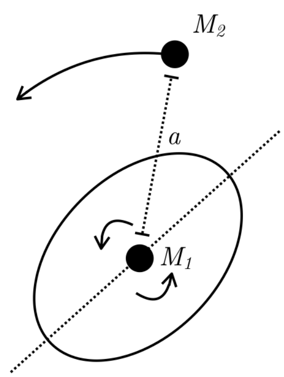

Figure 1: Diagram showing a point mass M_2 with a companion of mass M_1 with an exaggerated tidal bulge. The axis of the bulge is shown by a dotted line, which, in the case of co-rotation, would pass through M_2.

Consider a circular but unsynchronised orbit of a binary with component masses M_1 and M_2 and separation a. We denote the moment of inertia of M_1 by I_1, assumed to be constant, the spin angular momentum of M_1 by J_1, and the spin angular velocity of M_1 by Ω_1. We consider the simplifying case of the spin of M_2 being small, such that its moment of inertia may be neglected ($I_1\Omega_1 \gg I_2\Omega_2$).

Ignoring winds, the total angular momentum of the binary orbit is therefore

$$J = I_1\Omega_1 + \mu a^2 \Omega_{\text{orb}}, \tag{1}$$

where $1/\mu = 1/M_1 + 1/M_2$ and μ is the reduced mass.

We consider the scenario shown in Figure 1, where $\Omega_1 < \Omega_{\text{orb}}$, such that the bulge of M_1 lags behind M_2. Tidal torques act to transfer angular momentum from the orbit into the spin of M_1 by tightening the orbit. This increases Ω_1 to match Ω_{orb}. But with a tighter orbit, Ω_{orb} also increases. A natural question that arises is whether the binary ever reaches synchronisation:

- If $\dot{\Omega}_1 > \dot{\Omega}_{\text{orb}}$, the spin-up of M_1 is faster than the spin-up of the shrinking orbit. The system approaches synchronisation, $\Omega_{\text{orb}} - \Omega_1 \to 0^+$.
- If $\dot{\Omega}_1 < \dot{\Omega}_{\text{orb}}$, the shrinking orbit causes the orbit to spin up too quickly for M_1 to catch up, and $\Omega_{\text{orb}} - \Omega_1$ increases catastrophically. This implies the separation shrinks

catastrophically (on a dynamical time), ending with a merger. This is called the *Darwin instability.*

Claim 1. *With this setup, we have* $J_1 \frac{\dot{\Omega}_1}{\Omega_1} = \frac{1}{3} J_{orb} \frac{\dot{\Omega}_{orb}}{\Omega_{orb}}$.

Proof. By conservation of total angular momentum J,

$$0 = \dot{J} = \underbrace{I_1 \dot{\Omega}_1}_{=J_1 \frac{\dot{\Omega}_1}{\Omega_1}} + 2\mu a \dot{a} \Omega_{\rm orb} + \underbrace{\mu a^2 \dot{\Omega}_{\rm orb}}_{J_{\rm orb} \frac{\dot{\Omega}_{\rm orb}}{\Omega_{\rm orb}}}. \tag{2}$$

In the second term, rewrite $\dot{a}$ in terms of $\dot{\Omega}_{\rm orb}$ by noting that $\dot{\Omega}_{\rm orb}/\Omega_{\rm orb} = -(3/2)(\dot{a}/a)$. We then obtain the required expression by collecting like terms. □

A Corollary of the Claim is that requiring $\dot{\Omega}_1 > \dot{\Omega}_{\rm orb}$ for stability gives the stability criterion on the angular momenta:

$$\boxed{\text{The system is Darwin stable} \iff J_{\rm orb} > 3J_1,} \tag{3}$$

i.e. A synchronised orbit is achieved as long as M_1 has sufficiently small spin (less than a third of the orbit). The stability criterion may also be recast as a condition on the separation:

$$\boxed{\text{The system is Darwin stable} \iff a > a_{\rm Darwin} = \sqrt{\frac{3I_1}{\mu}}.} \tag{4}$$

Application to Binary Orbits

A binary that starts off satisfying Equation 4 may eventually shrink below $a_{\rm Darwin}$ due the evolution of M_1. Because $I_1 \sim M_1 R_1^2$, the radial expansion of M_1 during its evolution may cause $a_{\rm Darwin}$ to grow and supersede a, leading to the Darwin instability. However, if M_1 fills its Roche lobe before this occurs, then the Darwin instability never becomes relevant. Therefore, a binary never becomes Darwin unstable if its separation exceeds the maximally-realisable $a_{\rm Darwin}$, which is $a_{\rm Darwin}$ evaluated at the moment M_1 fills its Roche lobe, $x_L(q)a$: $a > a_{\rm Darwin}(R_1 = x_L(q)a)$, where $q = M_1/M_2$ is the mass ratio. Defining $I_1 = kM_1R_1^2$, this becomes

$$\frac{1}{3}\frac{a^2}{1+q} > k x_L(q)^2 a^2. \tag{5}$$

The separation a^2 cancels out, so we find a criterion that is independent of a and only depends on q and k. We may use the Eggleton (1983a) analytical approximation of the Roche radius,

$$x_L(q) = \frac{0.49q^{2/3}}{0.6q^{2/3} + \ln(1+q^{1/3})}. \tag{6}$$

Solving Equation 5 numerically, we find the following critical mass ratios:

- For a $n = 3$ polytrope, which is a reasonable description of a main sequence star, it may be shown that $k = 0.076$ (approximately a fifth that for a uniform sphere), upon which one finds the critical "Darwin" mass ratio $q_D \approx 12$, above which the binary encounters the Darwin instability and plunges in on a dynamical time. Tides do not have enough time to synchronise and circularise the orbit, meaning the binary may proceed to Roche-Lobe overflow with an unsynchronised and eccentric orbit. The likely outcome is merger.

- For a $n = 3/2$ polytrope, a reasonable description of a red giant or the convective core of a giant, we find $q_D \approx 5$.

We have neglected the spin of M_2 and assumed circularity. Hut [1980] performs an analysis for arbitrary eccentricity and includes M_2's spin, leading to the very similar stability criterion

$$\boxed{\text{The system is Darwin stable} \iff J_{\text{orb}} > 3J_1 \iff a > a_{\text{Darwin}} = \sqrt{\frac{3(I_1 + I_2)}{\mu}}} \tag{7}$$

References

P. Hut. Stability of tidal equilibrium. *Astronomy and Astrophysics*, 92(1-2):167–170, Dec 1980.