The Viscosity Problem and The Magnetorotational Instability

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1 The Viscosity Problem

The process of accretion requires angular momentum transport. Without angular momentum transport, a fluid element in a Keplerian accretion disk simply maintains its orbit with fixed specific angular momentum $J_K = \sqrt{GMr}$ and therefore fixed radius r. To move inward towards the central source, angular momentum of the fluid element must either be removed from the disk by an external torque or redistributed within the disk by an internal torque. This is equivalent to the inclusion of a source of viscous stress $\boldsymbol{\sigma}$ in the fluid momentum equation:

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \mathbf{F}, \tag{1}$$

where d/dt is the convective derivative and $\mathbf{F}$ accounts for any other body force such as gravity. $\boldsymbol{\sigma}$ is the stress tensor, which differs from the momentum flux tensor by a sign, $\boldsymbol{\sigma} = -\mathbf{\Pi}$. The theory of gas kinetics gives the following expression for $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = \eta\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{u}\right) + \zeta\mathbf{I}\nabla\cdot\mathbf{u}. \tag{2}$$

The stress tensor is a linear combination of velocity gradients. The first term is a traceless part that gives rise to *shear viscosity*, whose effect is to transport momentum in a direction orthogonal to the momentum. The proportionality constant η is the dynamic viscosity and is defined by this equation. The second term contains the trace and gives rise to bulk viscosity, whose effect is to transport momentum in the direction along it. The proportionality constant ζ is the bulk viscosity. The values of the viscosities η and ζ depend on microphysics. For an incompressible fluid, $\nabla \cdot \mathbf{u} = 0$; there is no bulk viscosity and the viscous stress tensor is symmetric and traceless. If η is also constant, Equation 1 is called the Navier-Stokes equation.

In an accretion disk, the main inertial flow is in the azimuthal direction and accretion requires angular momentum transport in the outward radial direction. This therefore requires a source of shear viscosity. An obvious and natural source choice to consider is molecular viscosity, where viscous dissipation arises from molecular collisions or friction. We will show, however, that the effects of molecular viscosity in accretion disks are very small and cannot match accretion rates inferred from observations.

1.1 Molecular Viscosity

An elementary kinetic theory calculation gives the kinematic viscosity $\nu = \eta/\rho = \frac{1}{3}\tilde{v}\tilde{\lambda}$, where ρ is the mass density of the fluid, $\tilde{v}$ and $\tilde{\lambda}$ are the unspecified velocity and length scales associated with the fluctuations in the fluid that give rise to the momentum transport, and the factor of 1/3 arises from averaging a particle velocity component over an isotropic velocity distribution. The kinematic viscosity therefore has scaling $\nu \sim \tilde{v}\tilde{\lambda}$.

The Reynolds number Re is given by the ratio of the inertial terms $\sim |\rho\partial\mathbf{u}/\partial t|$ to the viscous terms $\sim |\nabla\cdot\boldsymbol{\sigma}|$ in the momentum equation (1). The former has scaling

$$|\nabla\cdot\boldsymbol{\sigma}| \sim \frac{1}{R}\eta\frac{dv_\phi}{dr} \sim \frac{1}{R}\rho\tilde{v}\tilde{\lambda}\frac{v_\phi}{R} \sim \frac{v_\phi}{R^2}\rho\tilde{v}\tilde{\lambda}, \tag{3}$$

and the latter has scaling

$$\left|\rho\frac{\partial\mathbf{u}}{\partial t}\right| \sim \rho\frac{v_\phi^2}{R}. \tag{4}$$

Taking the ratio, we obtain

$$\boxed{Re \sim \frac{v_\phi}{\tilde{v}}\frac{R}{\tilde{\lambda}}} \tag{5}$$

For molecular viscosity, the relevant velocity and length scales are the sound speed $\tilde{v} \sim c_s$ and the thermal mean free path $\tilde{\lambda} \sim \lambda_{\mathrm{mfp}}$, and so

$$Re_{\mathrm{mol}} \sim \frac{v_\phi}{c_s}\frac{R}{\lambda_{\mathrm{mfp}}}. \tag{6}$$

For hydrogen plasma, $\nu_{\mathrm{mol}} \sim c_s\lambda_{\mathrm{mfp}} \sim 10^5\mathrm{cm}^2 s^{-1}$. Using $v_\phi = \sqrt{GM/R}$, $M \sim 10M_\odot$, and $R \sim 10^{10}$cm for accretion disks around close black hole binaries, $Re_{\mathrm{mol}} \gtrsim 10^{14}$ and so the accretion flow is almost completely uninfluenced by molecular viscosity. The factor v_ϕ/c_s is just the disk Mach number, $\mathcal{M}$, and cannot greatly exceed unity (a supersonic disk may have $\mathcal{M} \sim 10-20$). Thus, the largeness of Re_{mol} lies in the factor R/λ_{mfp}: the length scale of the accretion flow is so much larger than the scale at which molecular viscosity operates.

On the other hand, we know that turbulence onset occurs at the critical Reynolds numbers of order $10 - 10^3$, and so we expect an accretion disk flow to be turbulent. Then, the fluctuations required for momentum transport may be provided by turbulent eddies with characteristic velocity $\tilde{v}$ and characteristic size $\tilde{\lambda}$. Then, the Reynolds number associated with this turbulent viscosity may be small for the largest and fastest turbulent eddies as seen from (5).

1.2 Angular Transport by Turbulence

We now show that turbulence gives rise to a stress $\boldsymbol{\tau} = -\rho\langle\tilde{\mathbf{u}}\tilde{\mathbf{u}}\rangle$, called the Reynolds stress or turbulent stress, where $\tilde{u}$ is the turbulent fluctuation in the flow velocity.

We start by examining the momentum equation written in conservative form,

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) = -\nabla \cdot (\rho \mathbf{u}\mathbf{u}) - \nabla \cdot \mathbf{T} \tag{7}$$

where $\boldsymbol{\tau} = p\mathbf{I} - \boldsymbol{\sigma}$ is the (total) stress tensor, which includes momentum transport due to both viscosity and pressure. We decompose fluid quantities into a mean value plus a much smaller fluctuation due to turbulence: $\mathbf{u} = \mathbf{U} + \tilde{\mathbf{u}}$, $\boldsymbol{\tau} = \mathbf{T} + \tilde{\boldsymbol{\tau}}$. This is called a *Reynolds decomposition*. We assume that the mean quantities are constant with time and the fluctuations have zero ensemble average, $\langle \tilde{\mathbf{u}} \rangle = 0$ and $\langle \tilde{\boldsymbol{\tau}} \rangle = 0$, and that the fluid density ρ is constant. Carrying out the decomposition and subsequently taking the ensemble average, the equation becomes

$$0 = -\nabla \cdot (\mathbf{U}\mathbf{U}) - \frac{1}{\rho}\nabla \cdot (\mathbf{T} + \rho\langle \tilde{\mathbf{u}}\tilde{\mathbf{u}} \rangle). \tag{8}$$

Thus, the correlation of velocity fluctuations gives rise to a stress, $-\rho\langle \tilde{\mathbf{u}}\tilde{\mathbf{u}} \rangle$. In particular, for $i \neq j$ in $\rho\langle u_i u_j \rangle$, this is a source of shear turbulent viscosity.

Finding the source of the (turbulent) viscosity to explain observed accretion rates had been a central problem in accretion disk astrophysics for decades until the magnetorotational instability was identified by Balbus & Hawley (1991, 1998) as a possible source of turbulent viscosity.

2 Magnetorotational Instability (Balbus-Hawley Instability)

In the absence of a magnetic field, accretion disks satisfy the Rayleigh stability criterion and so are hydrodynamically stable. A particle executes retrograde epicycles about its mean circular orbit. MHD instability in an accretion disk is demonstrated by performing linear stability analysis of the incompressible ideal MHD equations for axisymmetric perturbations:

Continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{9}$$

Momentum equation: $$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0} - \rho \nabla \Psi \tag{10}$$

Magnetic induction equation: $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{11}$$

Energy equation: $$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right)\frac{p}{\rho^\gamma} = 0 \tag{12}$$

Incompressibility: $$\nabla \cdot \mathbf{u} = 0 \tag{13}$$

The ideal MHD equations neglect magnetic diffusivity (in the induction equation), and therefore assumes the magnetic Reynolds number is large $R_m \gg 1$. The momentum equation assumes the only source of stress is Maxwell stress, but we include a body force due to the gravitational potential Ψ of the central source. We have also included incompressibility explicitly. In a non-rotating fluid, linear MHD perturbations give rise to shear Alfvén waves and the fast/slow magnetosonic waves.

We proceed to consider linear perturbations of fluid quantities about their equilibrium values: $\rho = \rho_0 + \rho'$, $p = p_0 + p'$, $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}' = B_0\hat{\mathbf{z}} + \mathbf{B}'$, $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}' = r\Omega(r)\hat{\boldsymbol{\phi}} + \mathbf{v}'$. We assume axisymmetry and also assume a constant background magnetic field $B_0\hat{\mathbf{z}}$ in the direction of the disk normal. Without loss of generality, we write the perturbed quantities as a Fourier mode, e.g. $\rho' = \rho_0' e^{i(\omega t - k_z z - k_r r)}$, and further assume the wavenumber of the vertical perturbations to be much larger than that of radial perturbations for a thin disk, $|k_z| \gg |k_r|$. The resulting linearised equations, when solved, lead to the dispersion relation

$$\omega^4 - \omega^2\left(2k^2v_A^2 + \frac{d\Omega^2}{d\ln r} + 4\Omega^2\right) + k^2v_A^2\left(k^2v_A^2 + \frac{d\Omega^2}{d\ln r}\right) = 0 \tag{14}$$

where $v_A = B_0/\sqrt{\mu_0\rho_0}$ is the Alfvén speed. The MRI exists when $\omega^2 < 0$, which clearly requires

$$\boxed{k^2v_A^2 < -\frac{d\Omega^2}{d\ln r}.} \tag{15}$$

Again, for most physical disks, RHS > 0 as the angular velocity decreases outwards. Thus, if we allow k to have arbitrary size, there always exist a small enough wavenumber for which Inequality 15 is satisfied. Particularly, Keplerian disks are always unstable (with important caveats discussed later) to perturbations above a certain lengthscale. Noting $v_A \propto B_0$, the LHS of Inequality 15 informs us increasing the lengthscale of perturbations and decreasing the background magnetic field have a destabilising effect. The interpretation of this criterion is that an instability will set in if the magnetic tension, whose effect is to resist compression or rarefaction of field lines, is not strong enough to counteract the net tidal force (centrifugal minus gravitational force) acting on it.

When the instability criterion is satisfied, Equation 14 can be maximised to find the maximum growth rate

$$|\omega_{\max}| = \frac{1}{2}\left|\frac{d\Omega}{d\ln r}\right|. \tag{16}$$

Remarkably, this is independent of the field strength given it is non-zero. It is therefore incorrect to neglect even weak magnetic fields in accretion disk flow. For a Keplerian disk, $|\omega_{\max}| = \frac{3}{4}\Omega$ occurs at $kv_A = \frac{\sqrt{15}}{4}\Omega$. So the e-folding time of the instability is of order inverse angular velocity:

$$|\omega_{\max}|^{-1} \approx \Omega^{-1} \approx 90\mathrm{s}\left(\frac{R}{10^{10}\mathrm{cm}}\right)^{3/2}\left(\frac{M}{M_\odot}\right)^{-1/2}. \tag{17}$$

The requirement of a magnetic field for this instability can be understood as follows. In a non-magnetised fluid, a perturbed fluid element tends to conserve its specific angular momentum. When a fluid element is displaced radially outward, it has too little specific angular momentum for its new position, and so relaxes back towards its initial position. However, in ideal MHD, fluid flow entrains magnetic field lines, which enforce rigid rotation.

A perturbed fluid element tends to conserve its angular velocity instead. An outwardly-displace fluid element has too much angular velocity for its new position, and thus too much angular momentum for its new position, and is driven further outward. The magnetic field therefore provides a mechanism to couple the perturbed fluid element to the rest of the unperturbed fluid threaded by the same field line. In fact, the same magnetic torque that maintains isorotation of the perturbed fluid element must exert an equal and opposite torque to the inner fluid, which loses angular momentum and sinks closer to the centre.

It is also instructive to think of this from a dynamical perspective, using the Keplerian disk as an example. A particle on a circular Keplerian orbit is stable due to the balance of gravity and the centrifugal force. When perturbed, the fluid element conserves its angular momentum, and so while the gravitational force decreases as $\propto R^{-2}$, the centrifugal force decreases even more, as $\Omega^2 R \propto (J/R)^2 R \propto R^{-3}$. The net force is therefore directed towards the original orbit. The particle, due to its inertia, will overshoot. Following the same argument, it will accelerate towards its original orbit again, undergoing radial epicyclic oscillations about the circular orbit (this is just an elliptical orbit—the epicyclic frequency is equal to the orbital frequency of a Keplerian orbit). In a magnetised fluid, however, flux-freezing forces the displaced particle to conserve its angular velocity instead, and so the centrifugal force, going as $\Omega^2 R \propto R$, actually increases while the gravitational force weakens, causing runaway.

2.1 Caveats and Limitations

The claim that there always exists a small enough k for Inequality 15 to be satisfied for positive RHS is not strictly true, since the perturbation lengthscale is obviously limited by the disk scale height H. So for a thin disk, even if Inequality 15 is satisfied, the disk is stable if $v_A/\Omega > H$. For a thin disk, $\Omega H \sim c_s$, and so the disk is stable if $v_A > c_s$.

Another assumption is that for MRI to be applied to accretion disks, matter must be coupled to the magnetic field, i.e. there must be some degree of ionisation. This is, for example, not satisfied in protoplanetary disks where there is a very low ionisation fraction. In fact, the gravitational instability is the main source of viscosity in protoplanetary disks.

The other caveat stems from the assumption of ideal MHD, which neglects magnetic diffusivity η_B in the induction equation. However, it is possible for magnetic diffusivity to suppress an instability if they act on a similar lengthscale. Recalling that η_B is the diffusion coefficient for $\mathbf{B}$, the timescale for magnetic diffusion over a lengthscale $1/k$ is $\tau_{\text{diffusion}} \sim 1/(\eta_B k^2)$. On the other hand, the timescale of the MRI was previously shown to be $\tau_{\text{growth}} \sim 1/\Omega$. So we expect magnetic diffusivity to stabilise the disk against the MRI if $\tau_{\text{growth}} \gtrsim \tau_{\text{diffusion}} \implies \eta_B \gtrsim \Omega/k^2$, i.e. the magnetic Reynolds number is not large, $Re_m \lesssim 1$. This makes sense intuitively: large magnetic diffusivity and small perturbations favour the suppression of the MRI. But even then, the instability is present at large lengthscales. In fact, linear stability analyses including the effects of resistivity, ambipolar diffusion, and the Hall effect show the instability to be present for many non-ideal astrophysical plasmas.

Numerical simulations of magnetised accretion disks confirm the presence of the MRI, which is found to transport angular momentum outward in the disk. The dynamo action of the accretion disk is also found to regenerate the magnetic field to sustain this instability.

2.2 Application to the Shakura-Sunyaev Disk Model

Continuing from the discussion of turbulent viscosity, a turbulent disk may be described by a Reynolds stress $-\Sigma\langle \tilde{u_r}\tilde{u_\phi}\rangle$. We associate this with a shear viscosity ν by

$$-\Sigma\langle \tilde{u_r}\tilde{u_\phi}\rangle = \Sigma\nu r\frac{d\Omega}{dr}. \tag{18}$$

For a Keplerian disk, this gives $\langle \tilde{u_r}\tilde{u_\phi}\rangle \sim \nu\Omega$. Using $\nu = \alpha c_s H$ in the Shakura-Sunyaev prescription and $H \sim c_s/\Omega$ in a thin disk,

$$\alpha \sim \frac{\langle \tilde{u_r}\tilde{u_\phi}\rangle}{c_s^2}, \tag{19}$$

which gives the expression for α in disk accretion driven by shear turbulence. Balbus & Papaloizou (1999) point out that as long as the velocity correlation $\langle \tilde{u_r}\tilde{u_\phi}\rangle$ is positive, the magnetic turbulence may indeed be treated as an effective viscosity. If it also gives rise to a constant α, Equation 19 formally closes the Shakura-Sunyaev disk equations and solves the viscosity problem.